Stiefel-Whitney Classes and Unoriented Cobordism

Dylan Galt

In a Tortoiseshell: *In the* ***introduction*** *to his junior paper in mathematics, Dylan Galt presents the* ***motivating*** *questions underlying his exposition of Stiefel-Whitney classes and cobordism, preparing the reader to appreciate the significance of the rich mathematics which will follow.*

Excerpt

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This paper has two focuses: first, to give an introduction to Stiefel-Whitney classes; second, to describe some of their applications to the theory of cobordism. We will later on present the Stiefel-Whitney classes as unique cohomology classes satisfying a series of axioms (as done in Milnor and Stasheff), but will not prove that these axioms suffice to determine the existence and uniqueness of the classes which satisfy them. Therefore, for the reader, it will be helpful to have an understanding of why cohomology classes satisfying these axioms might be desirable, and it is in this spirit that we give the following overview of characteristic classes, of which the Stiefel-Whitney version are a special case.

Our particular aim in this brief introduction is to present a series of motivating questions, such that when the rigorous definitions are made later on, their importance, usefulness, and context will be more apparent. That said, what we present here is of course only an introduction; terminology and constructions mentioned will be elucidated in full later on. Consider the following set-up. Let X be any topological space and ξ a vector bundle over it. One might pose the following question:

To what extent does the topology of X control the complexity of ξ?

This is an open-ended question, but it is in many ways a rather natural one. As we will see, vector bundles are strict structures; their definition encodes a series of topological constraints on the relationship of total space to base space. What one wants essentially is a total space which looks locally like a product of a patch of the base space and a copy of Euclidean space. The closer the patch of the base space is to being the entire base space, the more "trivial? the bundle is, looking more like a global, rather than local, product. Therefore, one might indeed expect the triviality of the bundle ξ to depend on the relationship of the base space X, in its entirety, to its individual patches; in other words, the topology of X.

Let us consider, then, a more specific set-up, and ask the same question in a different and more approachable way. Let X be any topological space and ξ_1, ξ_2 two vector bundles over it. Consider the cohomology ring $H^*(X)$ of X. One might pose the following question:

Are there classes $c_i(\xi_1), c_i(\xi_2) \in H^i(X)$, for each index i, such that if ξ_1 and ξ_2 are isomorphic bundles, $c_i(\xi_1) = c_i(\xi_2)$ for each index i?

In other words, can one find distinguished classes in the cohomology of the base space of a vector bundle which detect the isomorphism class of that vector bundle? This question is still far too general to make sense of for every possible topological space. However, we will see that for a wide class of paracompact topological spaces, we will be able not only to answer our question in the affirmative, but in fact find a universal space equipped with a canonical bundle in such a way that our distinguished cohomology classes will all be pullbacks of classes in this universal space.

After presenting the necessary background, this paper will present two key theorems. Put together, these results will allow us to perform a series of calculations characterizing unoriented cobordism for 3 and 4-manifolds. These calculations will be the culmination of our work, displaying the surprising and captivating power of characteristic classes.

Author Commentary

Dylan Galt

My work on this junior paper began almost a year before I actually sat down to write it. In the summer before my junior year, I attended a conference at Princeton on symplectic geometry and low-dimensional topology, where I met Professor Szabó and developed an interest in his work. When my junior year began, I was unsure whether my time would be better spent pursuing research or building my foundations and initially I approached Szabó with an interest in working on an open problem that had been introduced at the conference. We resolved to meet weekly and over the course of the first half of the fall semester, we gradually settled towards building my foundations. This was a challenging period for me, as I suspect it is for many students eager for research results; it took me a while to realize the importance of progressing methodically. To this end, Szabó impressed upon me the value of learning theory well, not just of gaining exposure to it. Although I did not learn much mathematics in these first weeks, I learned quite a lot about what I did not know. I also got to know Professor Szabó and our acquaintance made it much easier for him to suggest reading that was appropriate for me.

Ultimately, we decided to work through Milnor and Stasheff's *Characteristic Classes*. I am now of the opinion that focusing oneself on a classic text and truly absorbing it is one of the best things one can do to gain mathematical maturity. The text presents various different characteristic classes, distinguished cohomology classes that detect the complexity of vector bundles, help to characterize unoriented cobordism, and give dimensional bounds for embeddings of real projective space into Euclidean space. When it came time to narrow the scope of what I had read to what would be appropriate for a junior paper, Szabó recommended that I select several chapters from Milnor and Stasheff, for which I would aim to summarize the material and do some computations of my own. To ensure that this presentation was coherent, I decided to focus on applications of Stiefel-Whitney classes to unoriented cobordism, computing the unoriented cobordism groups in three and four dimensions, and presenting theorems from a few additional resources to bolster the account. The bulk of the writing was a presentation of those theorems that would make possible the computations I wanted to give at the end, so that the paper would culminate in a series of concrete results.

Several choices were made in the course of writing my exposition, with the intent of pitching it towards the relatively uninitiated reader. Szabó had given me the excellent advice of writing the account that I thought would have been most useful and readable to the student I was nine months before, when the project had begun. With this in mind, I decided to devote the exposition to two things: first, motivating the construction of characteristic classes, and then, explaining the ultimate goal of the paper, namely the cobordism computations that were to cap it off. When I first approached Milnor and Stasheff, I was constantly asking myself why certain choices were made, why the theory had developed the way it did, and why one might have chosen to construct characteristic classes the way in which they were constructed. I focused the motivating component of my exposition on trying to explain these questions. In a similar spirit to Szabó's suggestion, I believe there is little else more frustrating in mathematics than an entirely unmotivated definition, and it was as such my express intent to give useful and hopefully insightful context. I also believe that sometimes the best moments in the study of mathematics are when, in the course of a lecture, the lecturer steps away from the blackboard, reverently, and explains that they will now discuss a bit of philosophy. After all, it is often in these moments that one might say the true meaning of a certain bit of mathematics becomes apparent. It is also for this reason that I felt a slightly longer, less rigorous, but more philosophical introduction was warranted.

Perhaps inevitably, one wonders what the content added is in an expository paper. Why should I write this if most of what I am doing is regurgitating definitions and re-proving theorems? There are two optimistic ways that I see to answer this question. The first is to consider the value of exposition to your own development: writing out proofs in your own words is a wonderfully effective means of remembering them. The second lies in those moments of exposition when you get a chance to present an insight. This is where you add unique value to a collection of technical material that could otherwise be found elsewhere. Just as solutions to large problems can sometimes be prompted by initially small revelations, so it is that a small reformulation, rewording, improved presentation, hint at larger context, or insight into meaning can make the difference in a reader's understanding. It is ultimately for this reason that I think the exposition is of particular importance in a mathematics paper.

Editor Commentary

Isabella Khan

Writing an introduction is tricky, especially for a longer work, which necessarily has a greater number of moving parts. In fact, this task is no less difficult when the paper is expository rather than research-based, as is often the case in undergraduate writings in mathematics. What are the goals of the introduction to an expository paper? More broadly, what is the aim of the expository paper itself? Dylan answers this latter question in his commentary, above: while an expository paper contains no new mathematics, he says, it can still present new analogies, which can then be used to create new mathematics in a different context. It is, therefore, the rearrangement of ideas according to the author's priorities, that is of value.

The introduction is the author's single greatest opportunity to present these priorities to the reader, and the excerpt of Dylan's junior paper printed above presents an excellent example of how this can be done. In this first section, Dylan introduces two key classes of objects, namely vector bundles and characteristic classes, framing his discussion in terms of the motivating questions which guide his entire exposition. By structuring his introduction in this way, he not only provides the reader with a roadmap to his paper, but also clarifies what new information he has added.

As Dylan points out in his commentary, while the motivating questions behind his exposition are clearly presented in the introduction, they were not as clear during the process of research and writing. Indeed, it is often only late in the process that we are able to fully understand the priorities underlying our exposition. The trick of writing a compelling introduction is to uncover these priorities and shift from whatever ordering of ideas arose naturally during the research process to an ordering that reflects the author's final conception of the piece as a whole. Dylan's introduction is remarkable in that is able to condense a long and circuitous research process into a succinct passage in which he orients the author to the priorities of his subject matter while also giving a taste of the pay-off he will discuss at the very end. Not only does he prepare his reader to appreciate the rich mathematics that will follow: from the very beginning of his paper, Dylan gives the reader the flavor of the subject, presenting his own perspective on the material and giving an insights into an already beautiful subject.

Bios

Dylan Galt '20 is a math major. He grew up in Mumbai, Beijing, Shanghai, Guangzhou, Virginia, Brussels, and Ulaanbaatar. His current favorite place to be is Wende Lu in Taipei, Taiwan. He loves piano composition and the first movement of Beethoven's fourth symphony, Carlos Kleiber conducting. Mathematically, his tastes are inclined towards homology theories, singular spaces, reductive algebraic groups, and—of course—characteristic classes. He wrote this as a senior.

Isabella Khan '21 is a junior in the mathematics department. She is originally from Chicago, although she would probably prefer to be from somewhere further West (Montana would do, but Oregon would be better). In her free time, she runs, plays violin, and regales her unsuspecting friends and relatives with all the facts about Middle-Earth that are floating around her head instead of schoolwork. She wrote this as a junior.